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The irreducible soluble subgroups of GL(3,3) and GL(3,5) and GL(5,3)

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ABSTRACT: In this paper we determine the imprimitive soluble subgroups of **GL(3,3) , GL(3,5)** and **GL(5,3)** .And a generating set for a JS-primitive of **GL(3,pk)** .Therefore we obtain a generating set for JS-primitive of **GL(3,pk) or** M_3**(3,p^k)** from the above table and so irreducible soluble subgroups of M_3**(3,p^k)**.

***Keywords*:** soluble subgroups, imprimitive subgroups, irreducible subgroups.

INTRODUCTION

Jordan(1871a)gives a table containing the numbers of conjugacy classes of primitive maximal soluble groups of degree lees then 10^6. He claims there are five such classes of degree 81，but there are only four，(see to Chapter 6).This error is Likely to lead to errors for larger degrees.Also，the Second And third entries in the last row of this table should be Swapped.In The same paper，Jordan also gives a table containing the numbers of conjugacy classes of transitive maximale soluble groups of degree Up to 10000.(This is an a stounding achievment if for no other reason Than the amount of counting required to prepare it.for example，if p is a prime greater than 3，Jordan claims that the number of conjugacy classes of transitive maximal soluble groups of degree $2^6 3^2 p$ is 8306).Again，the error in the first table is Likely to errors in this on too.Jordan (1872)caunts all the primitive groups of degrees 4 to 17.His count Matches that of sims except that Jordan has one les for degrees 9，12 and 15， eight less for degree 16 and two less For degree 17.These errors are Pointed out by Miller(1894，4b，1895c，1897a，1897b and 1900a).Jordan(1874)states that every transitive group of degree 19 is either alternating，symetric or affine.This agrees With sim's list.

Now in this paper we determines the irreducible soluble subgroups of $GL(3,3)$ and $GL(3,5)$ and $GL(5,3)$.
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Chapter 1:

In this chapter we paied a elemamntry definitions and basic theorems.
The elemantry definitions and theorems.

1.1 definition:

Let G，N and H are groups and G has a normal subgroup N_0 isomorphic to N such that $\frac{G}{N_0}$ is isomorphic to H，Then we write $G = N \lambda H$.

If G has a subgroup isomorphic to H which intesects N_0 trivially then G is a semidirect product of N and H，and we write $G = N \succ H$.

1.2 Definition:

We say that a group G has a central decomposition $(H_1,...,H_n)$ if

1 - each H_i is a normal subgroup of G.

2 - $G = H_1...H_n$.

3 - for each i and j, $H_i \cap H_j \leq Z(H_i) \cap Z(H_j)$

4 - for each i and j, $H_i \cap H_j$ *equals* $Z(H_i)$ *or* $Z(H_j)$

We also say that G is the central product of the H_i and $G = H_1 \curlyvee ... \curlyvee H_n$.

1.3-Definition:

The holomorph of group G, written $HOL(G)$, is the semidirect product of G and its automorphism group.

1.4-Definition:

We say that a group is monolithic if it has a unique minimal normal subgroup.

Chapter 2: Preliminaries

Let M be the JS-imprimitive of $GL(3,3)$, that is, $M := GL(1,3)\ wr\ S_3$.
This group has order 48, and is generated by the matrices

$$a := \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, b := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } c := \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In order to obtain a polycyclic presentation for M we introduce the element

$$d := c^b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \ \ and \ \ e := d^b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Then a polycyclic presentation for M is

$$\{a, b, c, d, e \mid \ a^2 = I_3 \ ,$$
$$b^a = b^2 \ , b^3 = I_3,$$
$$c^a = d \ \ , c^b = d \ \ , c^2 = I_3 \ ,$$
$$d^a = c \ \ , d^b = e \ \ , d^c = d, d^2 = I_3,$$
$$e^a = e \ \ , e^b = e \ \ , e^c = e \ \ , e^d = e \ \ , e^2 = I_3\}.$$

Note that $M = < cde > \times < a,b,ce,de > \cong C_2 \times S_4$.

2.1-Theorem B*:*

A complete and irredundant list of $GL(3,5)$ – conjugacy class representatives of the irreducible subgroups of M is:

$$
\begin{aligned}
&< a,b,c,d,e >, && \cong C_2 \times S_4;\\
&< a,b,ce,de >, && \cong S_4;\\
&< acde,b,ce,de >, && \cong S_4;\\
&< b,c,d,e >, && \cong C_2 \times A_4\\
&< b,ce,de >, && \cong A_4.
\end{aligned}
$$

proof:

See short (1992, 6.2, pp.77-78)

Now let M be the JS-imprimitive of $GL(3,5)$, that is ,

$M := GL(1,5)\ wr\ S_3.$

This group has order 384, and is generated by the matrices

$$a := \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, b := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } c := \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In order to obtain a polycyclic presetation for M we introduce the elements

$$d := c^b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } e := d^b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Then a polycyclic presentation for M is

$\{a,b,c,d,e \mid a^2 = I_3\ ,$

$$
\begin{aligned}
&b^a = b^2, b^3 = I_3,\\
&c^a = d\ , c^b = d, c^4 = I_3,\\
&d^a = c\ , d^b = e, d^c = d, d^4 = I_3,\\
&e^a = e\ , e^b = c, e^c = e, e^d = e, e^4 = I_3\}
\end{aligned}
$$

Note that

$$
\begin{aligned}
M &= < cde > \times < a(cde)^2, b, ce^{-1}, de^{-1} >\\
&= < cde > \times (M \cap SL(3,5)).
\end{aligned}
$$

2.2-Theorem C:

A complete and irredundant list of $GL(3,5)$ - conjugacy class representatives of the irreducible subgroups of M is:

$< a(cde)^2, b, ce^{-1}, de^{-1}, (cde)^i >, i = 1,2,4;$
$< a(cde)^{2+j}, b, ce^{-1}, de^{-1} >, j = 1,2;$
$< b, ce^{-1}, de^{-1}, (cde)^k >, k = 1,2,4;$
$< a(cde)^2, b, (ce^{-1})^2, (de^{-1})^2, (cde)^l >, l = 1,2,4;$
$< a(cde)^{2+m}, b, (ce^{-1})^2, (de^{-1})^2 >, m = 1,2;$
$< b, (ce^{-1})^2, (de^{-1})^2, (cde)^n >, n = 1,2,4.$

Proof:

Set B:= $< cde > \text{ and } N := M \cap SL(3,5).$ We use the CAYLEY function lattice to obtain the subgroup lattice of N. If subgroup of N is irreducible, then its projection into the top group must be transitive. Therefore the 2-subgroups of N are reducible.This leaves just six conjuacy classes of subgroups to consider, the groups in those classes being isomorphic to one of N, 48.52(this is the number of that group in the tables of Neubuser (1967), S_4, A_4, S_3 or C_3.

It is easly to show, then that the only irreducible subgroups of N are N, 48.52, S_4 and A_4. The proof now proceeds in the same way as that of

Theorm 4.3.4. Note that none theses groups a normal subgroups of index 4, and that 48.52 has no subgroups of index 2.The relevant table is table 1.

Table1. Information on the imprimitive soluble subgroups of $GL(3,5)$

G	$\|G\|$	$G \cap SL(3,5)$	$\|G \cap B\|$
$< a(cde)^2, b, ce^{-1}, de^{-1}, (cde)^i >, i = 1,2,4$	$\frac{384}{i}$	N	$\frac{4}{i}$
$< a(cde)^{2+j}, b, ce^{-1}, de^{-1} >, j = 1,2$	$\frac{192}{j}$	48.52	$\frac{2}{j}$
$< b, ce^{-1}, de^{-1}, (cde)^k >, k = 1,2,4$	$\frac{192}{k}$	48.52	$\frac{4}{k}$
$< a(cde)^{2+m}, b, (ce^{-1}), (de^{-1})^2 >, m = 1,2;$	$\frac{48}{m}$	A_4	$\frac{2}{m}$
$< b, (ce^{-1})^2, (de^{-1})^2, (cde)^n >, n = 1,2,4;$	$\frac{48}{m}$	A_4	$\frac{4}{n}$
$< a(cde)^2, b, (ce^{-1})^2, (de^{-1})^2, (cde)^l >, l = 1,2,4$	$\frac{96}{l}$	S_4	$\frac{4}{l}$

Now for determine the imprimitive soluble subgroups of $GL(5,3)$, let M be the JS -imprimitive of $GL(5,3)$, that is, $M := GL(1,3) \; wr \; Hol(C_5)$.

By the same methods as in the previous two case we can write down a polycyclic presentation for M, is:

$M = \{a, b, c, d, e, f, g \mid a^4 = I_5,$

$$b^a = b^2, b^5 = I_5,$$
$$c^a = d \ , c^b = d, c^2 = I_5,$$
$$d^a = f, d^b = e \ , d^c = d, d^2 = I_5,$$
$$e^a = c \ , e^b = f \ , e^c = e \ , e^d = e \ , e^2 = I_5,$$
$$f^a = e, f^b = g \ , f^c = f, f^d = f, f^e = f, f^2 = I_5,$$
$$g^a = g, g^b = c \ , g^c = g, g^d = g \ , g^e = g, g^f = g \ , g^2 = I_5\}.$$

Note that $M =< cdefg > \times < acdefg, b, cd, de, ef, fg >=< cdefg > \times (M \cap SL(5,3))$

2.3-Theorem D:

A complete and irredundant list of $GL(5,3)$ - conjugacy class representative of the irreducible subgroups of M is:

$$< acdefg, b, cd, ef, fg, (cdefg)^i >, i = 1,2;$$
$$< a, b, cd, de, ef, fg >;$$
$$< a^2, b, cd, de, ef, fg, (cdefg)^j >, j = 1,2;$$
$$< a^2cdefg, b, cd, de, ef, fg >;$$
$$< b, cd, de, ef, fg, (cdefg)^k >, k = 1,2.$$

Proof:

Set $B :=< cdefg >$ and $N :=< a, b, cd, de, ef, fg >$. We use the CAYLEY function lattice to obtain the subgroup lattice of N.

If a subgroup of N is irreducible，then its projection into then its projection into the top group must be transitive. Therefore the 2-subgroups of N are

irreducible. This leaves just six conjugacy classes of subgroups to consider， the groups in those classes being isomorphic to one of N , $\frac{1}{2}N$ (meaning the unique subgroup of N of index 2)， $\frac{1}{4}N$ (meaning the uniqe subgroup of N of index 4)， $Hol(C_5), D_{10} \ or \ C_5$. It is easly to show，then，that the only irreducible subgroups of N are $N, \frac{1}{2}N \ and \ \frac{1}{4}N$ .The proof now proceeds in the same way as that of 4.3.4 Theorem A.The relevant table is Table 2.

Table 2. Information on the primitive soluble subgroups of $GL(5,3)$

G	$\lvert G \rvert$	$G \cap SL(5,3)$	$\lvert G \cap B \rvert$
$< acdefg, b, cd, ef, fg, (cdefg)^i >, i = 1,2$	$\frac{640}{i}$	N	$\frac{2}{i}$
$< a, b, cd, de, ef, fg >,$	320	$\frac{1}{2}N$	1
$< a^2, b, cd, de, ef, fg, (cdefg)^i >, j = 1,2$	$\frac{320}{j}$	$\frac{1}{2}N$	$\frac{2}{j}$
$< a^2cdefg, b, cd, de, ef, fg >$	160	$\frac{1}{4}N$	1
$< b, cd, de, ef, fg, (cdefg)^k >, k = 1,2$	$\frac{160}{k}$	$\frac{1}{4}N$	$\frac{2}{k}$

Chapter 3:

Main result

3.1- JS–maximals of $GL(q, p^k),$ for q=3,5,7 and $p^k = 1,...,30.$

We saw that JS-maximals of $GL(q, p^k), q > 2,$ as follows .

$$M_1(q, p^k) := GL(1, p^k) \ wr \ \ Hol_{(}C_{q)}, \ p^k \neq 2,$$
$$M_2(q, p^k) := C_{p^{kq}-1} \succ C_q \ ,$$
$$M_3(q, p^k) := (C_{p^k-1} \mathrm{Y} \ E) \ \mathrm{N} \ D.$$

Where E is extraspecial of order q^3 and exponent q, and D is a maximal irreducible soluble subgroup of $Sp(2, q).$ Therefore if $q = 3,5,7$.

Then the JS-maximals of $GL(q, p^k)$ as will follows.

$$M_1(3, p^k) = GL(1, p^k) \ wr \ S_3, p^k \neq 2,$$
$$M_2(3, p^k) = C_{p^{3k}-1} \succ C_3,$$
$$M_3(3, p^k) = (C_{p^k-1} \mathrm{Y} \ E_{27}) \ \mathrm{N} \ Sp(2,3) \ , p^k \equiv 1(\bmod 3).$$

$$M_1(5, p^k) = GL(1, p^k) \ wr \ Hol(C_5), p^k \neq 2,$$
$$M_2(5, p^k) = C_{p^{5k}-1} \succ C_5,$$
$$M_3(5, p^k) = (C_{p^k-1} \mathrm{Y} \ E_{125}) \ \mathrm{N} \ Sp(2,5) \ , p^k \equiv 1(\bmod 5).$$

$$M_1(7, p^k) = GL(1, p^k) \ wr \ Hol(C_7), p^k \neq 2,$$
$$M_2(7, p^k) = C_{p^{7k}-1} wr \ C_7,$$
$$M_3(7, p^k) = (C_{p^k-1} \mathrm{Y} \ E_{7^3}) \ \mathrm{N} \ Sp(2,7) \ , p^k \equiv 1(\bmod 7).$$

Thus we have the following table for $q = 3,5,7$ and $p^k = 1,...,30.$

$GL(3,1)$	M_1	M_2		$GL(5,1)$	M_1	M_2		$GL(7,1)$	M_1	M_2	
$GL(3,2)$		M_2		$GL(5,2)$		M_2		$GL(7,2)$		M_2	
$GL(3,3)$	M_1	M_2		$GL(5,3)$	M_1	M_2		$GL(7,3)$	M_1	M_2	
$GL(3,4)$	M_1	M_2	M_3	$GL(5,4)$	M_1	M_2		$GL(7,4)$	M_1	M_2	
$GL(3,5)$	M_1	M_2		$GL(5,5)$	M_1	M_2		$GL(7,5)$	M_1	M_2	
$GL(3,6)$	M_1	M_2		$GL(5,6)$	M_1	M_2	M_3	$GL(7,6)$	M_1	M_2	
$GL(3,7)$	M_1	M_2	M_3	$GL(5,7)$	M_1	M_2		$GL(7,7)$	M_1	M_2	
$GL(3,8)$	M_1	M_2		$GL(5,8)$	M_1	M_2		$GL(7,8)$	M_1	M_2	M_3
$GL(3,9)$	M_1	M_2		$GL(5,9)$	M_1	M_2		$GL(7,9)$	M_1	M_2	
$GL(3,10)$	M_1	M_2	M_3	$GL(5,10)$	M_1	M_2		$GL(7,10)$	M_1	M_2	
$GL(3,11)$	M_1	M_2		$GL(5,11)$	M_1	M_2	M_3	$GL(7,11)$	M_1	M_2	
$GL(3,12)$	M_1	M_2		$GL(5,12)$	M_1	M_2		$GL(7,12)$	M_1	M_2	

$GL(3,13)$	M_1	M_2	M_3	$GL(5,13)$	M_1	M_2		$GL(7,13)$	M_1	M_2	
$GL(3,14)$	M_1	M_2		$GL(5,14)$	M_1	M_2		$GL(7,14)$	M_1	M_2	
$GL(3,15)$	M_1	M_2		$GL(5,15)$	M_1	M_2		$GL(7,15)$	M_1	M_2	M_3
$GL(3,16)$	M_1	M_2	M_3	$GL(5,16)$	M_1	M_2	M_3	$GL(7,16)$	M_1	M_2	
$GL(3,17)$	M_1	M_2		$GL(5,17)$	M_1	M_2		$GL(7,17)$	M_1	M_2	
$GL(3,18)$	M_1	M_2		$GL(5,18)$	M_1	M_2		$GL(7,18)$	M_1	M_2	
$GL(3,19)$	M_1	M_2	M_3	$GL(5,19)$	M_1	M_2		$GL(7,19)$	M_1	M_2	
$GL(3,20)$	M_1	M_2		$GL(5,20)$	M_1	M_2		$GL(7,20)$	M_1	M_2	
$GL(3,21)$	M_1	M_2		$GL(5,21)$	M_1	M_2	M_3	$GL(7,21)$	M_1	M_2	
$GL(3,22)$	M_1	M_2	M_3	$GL(5,22)$	M_1	M_2		$GL(7,22)$	M_1	M_2	M_3
$GL(3,23)$	M_1	M_2		$GL(5,23)$	M_1	M_2		$GL(7,23)$	M_1	M_2	
$GL(3,24)$	M_1	M_2		$GL(5,24)$	M_1	M_2		$GL(7,24)$	M_1	M_2	
$GL(3,25)$	M_1	M_2	M_3	$GL(5,25)$	M_1	M_2		$GL(7,25)$	M_1	M_2	
$GL(3,26)$	M_1	M_2		$GL(5,26)$	M_1	M_2	M_3	$GL(7,26)$	M_1	M_2	
$GL(3,27)$	M_1	M_2		$GL(5,27)$	M_1	M_2		$GL(7,27)$	M_1	M_2	
$GL(3,28)$	M_1	M_2	M_3	$GL(5,28)$	M_1	M_2		$GL(7,28)$	M_1	M_2	
$GL(3,29)$	M_1	M_2		$GL(5,29)$	M_1	M_2		$GL(7,29)$	M_1	M_2	M_3
$GL(3,30)$	M_1	M_2		$GL(5,30)$	M_1	M_2		$GL(7,30)$	M_1	M_2	

We already determined the imprimitive soluble subgroups of $GL(3,3)$, $GL(3,5)$ and $GL(5,3)$.And a generating set for a JS-primitive of $GL(3,p^k)$.Therefore we obtain a generating set for JS-primitive of $GL(3,p^k)$, or $M_3(3,p^k)$,from the above table’and so’irreducible soluble subgroups of $M_3(3,p^k)$..

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